Adaptive Output Feedback Control of Spacecraft Relative Translation

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Abstract—We address the problem of tracking relative translation in a leader-follower spacecraft formation using position feedback and under parameter uncertainty (spacecraft mass) and uncertainty in the leader variables (true anomaly rate and rate of change). We only assume boundedness of orbital perturbations and the leader control force but with unknown bounds. Under these conditions we propose a controller that renders the closed-loop system weakly uniformly semiglobally practically asymptotically stable. In particular, the domain of attraction can be made arbitrarily large by picking convenient gains, and the state errors in the closed-loop system are proved to converge from any initial condition within the domain of attraction to a ball in close vicinity of the origin in a stable way; moreover, this ball can be diminished to a maximum precision by increasing the gains in the control law. Simulation results of a leader-follower spacecraft formation using the proposed controller are presented.

I. INTRODUCTION

A. Background

Spacecraft flying in formation are revolutionizing our way of performing space-based operations, and brings out several advantages in space mission accomplishment, as well as new opportunities and applications for such missions. The concept makes the way for new and better applications in space industry, and replacing large and complex spacecraft with a fleet of micro-satellites introduces a multitude of advantages regarding mission cost and performance. However, the advantages of using spacecraft formations come at a cost of increased complexity and technological challenges.

Formation flying introduces a control problem with strict and time-varying boundaries on spacecraft reference trajectories, and requires detailed knowledge and tight control of relative distances and velocities for participating spacecraft. The control problem for the follower simplifies as the knowledge about the leader and its orbit increases. Complete knowledge of spacecraft masses, orbital parameters and orbital perturbations is hard to achieve, since these often change during operation. In addition, velocity sensors are costly, heavy and computationally demanding, and therefore the follower spacecraft must often rely on measurements of the position of the leader spacecraft only. Hence, the challenge lies in synchronized control of the formation, with as little exchange of information between spacecraft as possible.

B. Previous Work

The first solution to the position feedback control problem in leader-follower spacecraft formations was presented in [1], and use of the nonlinear control law results in global uniform ultimate boundedness of position and velocity tracking errors. The solution includes a filtering scheme to allow for use of knowledge of relative velocity in the controller equations. A similar result was also presented in [2], providing the same stability properties. Nonlinear adaptive tracking control was developed in [3], a result which ensures global asymptotic convergence to zero, of the tracking errors only. This result relies on the restrictive assumption of a circular orbit. In [4], a nonlinear tracking controller for both translation and rotation was presented, including an adaptation law to account for unknown mass and inertia parameters of the spacecraft. The controller ensures global asymptotic convergence of position and velocity errors; the proof relies on a standard signal-chasing analysis and Barbalat’s lemma. Based on the latter two references, semiglobal asymptotic convergence of relative translation errors was proved in [5] for an adaptive output feedback controller using relative position only, with a similar filtering scheme as in [2]. This result was extended to a similar result for both relative translation and rotation in [6], tracing the steps of [4].

C. Contribution

The purpose of this study is to provide a solution to the spacecraft formation control problem with as little knowledge about the leader spacecraft as possible. We present a solution to the problem of tracking relative translation in a leader-follower spacecraft formation using feedback from relative position only. The solution is an extension of results presented in [7], and relies on measurements of relative position only. An additional filter, similar to the one in [8], is included, using the method of approximate differentiation, or “dirty derivatives”, to provide sufficient knowledge about the relative velocity to solve the control problem. In addition, adaptation is used to find follower spacecraft mass, together with constant approximations to orbital perturbations, leader true anomaly rate and rate of change, and leader control force. The resulting closed-loop system is proved to be weakly uniformly semiglobally practically asymptotically stable (USPAS), meaning that the domain of attraction can be made arbitrarily large by picking convenient gains, and the state errors in the closed-loop system are proved to converge from any initial condition within the domain of attraction to a ball in close vicinity of the origin in a stable way, and the radius of this ball can be diminished to a lower bound $\delta^*$ by increasing the gains in the control law.

The rest of the paper is organized as follows: Section II contains notation and mathematical preliminaries. In Sect. III the model of relative translation is presented. The tracking control law is presented in Sect. IV. In Sect. V we present simulation results using the proposed controllers. Finally, some conclusions are contained in Sect. VI.
II. Mathematical Preliminaries

In the following, we denote by \( \dot{x} \) the time derivative of a vector \( x \), i.e. \( \dot{x} = dx/dt \). Moreover, \( \ddot{x} = d^2x/dt^2 \). We denote by \( x(t, t_0, x_0) \) the solution to the nonlinear differential equation \( \dot{x} = f(t, x) \) with initial conditions \((t_0, x_0)\). We denote by \(|x|\) the Euclidean norm of a vector and the induced \( L_2 \) norm of a matrix. We denote by \( B_{\delta} \) the closed ball in \( \mathbb{R}^n \) of radius \( \delta \), i.e. \( B_{\delta} := \{ x \in \mathbb{R}^n \mid |x| \leq \delta \} \). For such a ball we denote

\[
|z|_{\delta} = \inf_{x \in B_{\delta}} |z - x|.
\]

A continuous function \( \alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is said to be of class \( K_c \) if it is strictly increasing and \( \alpha(0) = 0 \). Moreover, \( \alpha \) is of class \( K_\infty \) \( (\alpha \in K_\infty) \) if, in addition, \( \alpha(s) \rightarrow \infty \) as \( s \rightarrow \infty \). A continuous function \( \sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is of class \( L \) \( (\sigma \in L) \), if it is strictly decreasing and \( \sigma(s) \rightarrow 0 \) as \( s \rightarrow \infty \). When the context is sufficiently explicit, we may omit to write arguments of a function.

A. Stability with Respect to Balls

For a general nonlinear system

\[
\dot{x} = f(t, x)
\]

where \( f(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is locally Lipschitz in \( x \) and piecewise continuous in \( t \), we use the following definitions of robust stability. Let \( \delta \) and \( \Delta \) be positive numbers such that \( \Delta > \delta \) and let them generate closed balls \( B_{\delta} \) and \( B_{\Delta} \) as previously defined.

Definition 1 (US of a ball [9]) For the system (2) the ball \( B_{\delta} \) is said to be uniformly stable (US) on \( B_{\Delta} \) if there exists a class \( K_\infty \) function \( \alpha \) such that the solutions of (2) from any initial state \( x_0 \in B_{\Delta} \) and initial time \( t_0 \geq 0 \) satisfy

\[
|x(t, t_0, x_0)|_{\delta} \leq \alpha(|x_0|), \quad \forall t \geq t_0.
\]

Definition 2 (UA of a ball [9]) The closed ball \( B_{\delta} \) is said to be uniformly attractive (UA) on \( B_{\Delta} \) if there exists a class \( L \) function \( \sigma \) such that the solutions of (2) from any initial state \( x_0 \in B_{\Delta} \) and initial time \( t_0 \geq 0 \) satisfy

\[
|x(t, t_0, x_0)|_{\delta} \leq \sigma(t - t_0), \quad \forall t \geq t_0.
\]

Definition 3 (UAS of a ball [9]) For the system (2) the ball \( B_{\delta} \) is said to be uniformly asymptotically stable (UAS) on \( B_{\Delta} \) if \( B_{\delta} \) is both uniformly stable and uniformly attractive on \( B_{\Delta} \).

B. Semiglobal Practical Stability

For parameterized nonlinear systems of the form

\[
\dot{x} = f(t, x, \theta)
\]

where \( f(t, x, \theta) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is locally Lipschitz in \( x \) and piecewise continuous in \( t \) for all \( \theta \) under consideration, we have the following stability definitions:

Definition 4 (weak USPAS) Let \( \Theta \subset \mathbb{R}^m \) be a set of parameters. The system (5) is said to be weakly uniformly semiglobally practically asymptotically stable (USPAS) on \( \Theta \) if there exists \( \delta^* > 0 \) and, for any \( \Delta > \delta > \delta^* \), there exists \( \theta^* \in \Theta \) such that for \( x = f(t, x, \theta^*) \) the ball \( B_{\delta} \) is uniformly asymptotically stable on \( B_{\Delta} \).

In other words, we say that (5) is weakly USPAS if the set of initial states can be arbitrarily enlarged, and the set with respect to which the system is UAS can be diminished to a lower bound \( \delta^* \), by a convenient choice of the parameters \( \theta \). From Definition 4, if \( \delta^* = 0 \), we recover the notion of uniform semiglobal practical asymptotic stability (USPAS), and moreover, if \( \delta = 0 \), we recover the notion of uniform semiglobal asymptotic stability (USAS). Also, if the system has the property of Definition 4 with \( \Delta = +\infty \) we say that the ball \( B_{\delta} \) is weakly uniformly globally practically asymptotically stable (UGPAS). Finally, if \( \delta = \delta^* = 0 \) and \( \Delta = +\infty \) we recover the definition of uniform global asymptotic stability (UGAS). In a number of concrete control problems the parameter \( \theta \) corresponds to the values of the control gains.

For the stability analysis of our main result, we rely on the following result, presented earlier in [7]. The result, which is a corollary of [9, Proposition 2], applies to the semiglobal practical stability analysis of systems presenting a Lyapunov function that can be upper and lower bounded by a polynomial function, as this situation arises very often in concrete applications. Typically, the parameter \( \theta \) contains the gain matrices that can be freely tuned in order to enlarge and diminish the domain of attraction and the vicinity of the origin to which solutions converge. To fix the ideas, the functions \( \sigma_i(\theta) \) refer often to the minimal eigenvalues of these gain matrices. So, roughly speaking, we impose that the dependency of these minimum eigenvalues in the radii \( \Delta \) and \( 1/\delta \) be polynomial and of an order not greater that the one of the bounds on \( V \).

Corollary 5 Let \( \sigma_i : \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}, i \in \{1, \ldots, N\} \), be continuous functions, positive over \( \Theta \), and let \( g, \pi \) and \( q \) be positive constants. Assume that, for any \( \theta \in \Theta \), there exists a continuously differentiable Lyapunov function \( V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) satisfying, for all \( x \in \mathbb{R}^n \) and all \( t \geq 0 \),

\[
\sigma_i(\theta) \leq V(t, x) \leq \pi \max \{ \sigma_i(\theta) \} \|x\|^q.
\]

Assume also that there exists \( \delta^* > 0 \) such that, for any \( \Delta > \delta > \delta^* \), there exists a parameter \( \theta^*(\delta, \Delta) \in \Theta \) and a class \( K \) function \( \alpha_{\delta, \Delta} \) such that, for all \( |x| \in [\delta, \Delta] \) and all \( t \geq 0 \),

\[
\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x, \theta^*) \leq -\alpha_{\delta, \Delta}(|x|).
\]

Assume also that for all \( i \in \{1, \ldots, N\} \) and for every fixed \( \delta > \delta^* \),

\[
\lim_{\Delta \rightarrow \infty} \frac{\sigma_i(\theta^*(\delta, \Delta))}{\Delta^q} = 0 \quad \text{and} \quad \lim_{\Delta \rightarrow \infty} \sigma_i(\theta^*(\delta, \Delta)) \neq 0.
\]

Then, the system \( \dot{x} = f(t, x, \theta) \) is weakly USPAS on the parameter set \( \Theta \). Moreover, when \( \Delta = \infty \) and the parameter \( \theta^* \) is independent of \( \Delta \), i.e. \( \theta^* = \theta^*(\delta) \), the conditions in (8) are no longer required, and the system \( \dot{x} = f(t, x, \theta) \) is weakly UGPAS on the parameter set \( \Theta \).

It should be underlined that the assumptions of Corollary 5 are much more conservative than the ones of the original result [9, Proposition 2]. The previous statement has, however, the important advantage of being more easily applicable and fits to many concrete applications. An example to illustrate the use of Corollary 5 can be found in [7].

III. The Model

Having established the mathematical framework for our main result, let us formulate the spacecraft formation problem.
A. Coordinate reference frames

The coordinate reference frames used throughout the paper are defined as follows:

**Earth Centered Inertial (ECI) frame:** This frame is denoted \( F_i \), and has its origin located in the center of the Earth. Its \( z \) axis is directed along the rotation axis of the Earth towards the celestial north pole, the \( x \) axis is directed towards the vernal equinox, and finally the \( y \) axis completes a right-handed orthogonal frame.

**Leader orbit reference frame:** The leader orbit frame, denoted \( F_l \), has its origin located in the center of mass of the leader spacecraft. The \( e_x \) axis in the frame is parallel to the vector \( r_l \), pointing from the center of the Earth to the leader, and the \( e_h \) axis is parallel to the orbit momentum vector, which points in the orbit normal direction. The \( e_\theta \) axis completes the right-handed orthogonal frame. The basis vectors of the frame can be defined as

\[
e_r = \frac{r_l}{|r_l|}, \quad e_\theta = e_h \times e_r \quad \text{and} \quad e_h = \frac{h}{|h|},
\]

where \( h = r_l \times \dot{r}_l \) is the angular momentum vector of the orbit.

**Follower orbit reference frame:** This frame has its origin at the center of mass of the follower spacecraft, and is denoted \( F_f \). The vector pointing from the center of the Earth to the center of the follower orbit frame is denoted \( r_f \). Its origin is specified by a relative orbit position vector \( p = [x \ y \ z]^T \) expressed in \( F_l \) frame components, and the frame unit vectors align with the basis vectors of \( F_l \). Accordingly,

\[
p = r_f - r_l = xe_r + ye_\theta + ze_h.
\]

B. Relative Translational Motion

The general orbit equation for two point masses \( m_1 \) and \( m_2 \) (cf. [10])

\[
\ddot{r} + \frac{\mu}{r^3} r = 0 \tag{11}
\]

where \( r \) is the relative position of masses, \( r = |r| \), and \( \mu = G (m_1 + m_2) \), \( G \) being the universal constant of gravity, is the equation describing the uncontrolled orbit dynamics for a spacecraft under ideal conditions. This equation can be generalized to include force terms due to control input vectors from onboard actuators, aerodynamic disturbances, gravitational forces from other bodies, solar radiation, magnetic fields and so on. Accordingly, (11) can be expressed for the leader and follower spacecraft as

\[
\ddot{r}_l = -\frac{\mu}{r_l^3} r_l + \frac{f_{dl}}{m_l} + \frac{u_l}{m_l}, \tag{12}
\]

\[
\ddot{r}_f = -\frac{\mu}{r_f^3} r_f + \frac{f_{df}}{m_f} + \frac{u_f}{m_f}, \tag{13}
\]

where \( f_{dl}, f_{df} \in \mathbb{R}^3 \) are the disturbance force terms due to external perturbation effects and \( u_l, u_f \in \mathbb{R}^3 \) are the actuator forces of the leader and follower, respectively. In addition, spacecraft masses are assumed to be small relative to the mass of the Earth \( M_e \), so \( \mu = GM_e \). Taking the second order derivative of the relative position vector \( p = r_f - r_l \) and using the true anomaly \( \nu(t) \) of the leader, which is the orbit plane angle measured in the center of the Earth between the orbit perigee point and the leader center of mass, the relative position dynamics can be written as (cf. [11])

\[
m_f \ddot{p} + C(\dot{\nu}) \dot{p} + D(\dot{\nu}, \dot{\nu}, r_f) p + m_f u(r_l, r_f) = U + F_d \tag{14}
\]

where

\[
C(\dot{\nu}) = 2m_f \begin{bmatrix} 0 & -\dot{\nu} & 0 \\ \dot{\nu} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
D(\dot{\nu}, \dot{\nu}, r_f) = m_f \begin{bmatrix} \frac{\mu}{r_f^3} - \dot{\nu}^2 & -\dot{\nu} & 0 \\ \dot{\nu} & \frac{\mu}{r_f^3} - \dot{\nu}^2 & 0 \\ 0 & 0 & \frac{\mu}{r_f^3} \end{bmatrix}
\]

may be viewed as a time-varying potential force;

\[
n(r_l, r_f) = \mu \begin{bmatrix} r_l \times \frac{1}{r_f^3} & 0 \\ 0 & 0 & 0 \end{bmatrix} ;
\]

the composite disturbance force \( F_d \) and the relative control force \( U \) are given by

\[
F_d = f_{df} - \frac{m_f}{m_l} f_{dl} \quad \text{and} \quad U = u_f - \frac{m_f}{m_l} u_l.
\]

For control design, we introduce the more convenient notation

\[
C(\dot{\nu}) = 2m_f \dot{\nu} \ddot{C}, \quad D(\dot{\nu}, \dot{\nu}, r_f) = m_f \frac{\mu}{r_f^3} \bar{I} + m_f \dot{\nu} \ddot{D} + m_f \dot{\nu} \dddot{C} \tag{19}
\]

\[
\dddot{C} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dddot{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}
\]

The rate of the true anomaly of the leader spacecraft is

\[
\dot{\nu}(t) = \frac{n_l (1 + e_l \cos \nu(t))^2}{(1 - e_l^2)^{3/2}} \tag{22}
\]

where \( n_l = \sqrt{\mu/a_l^3} \) is the mean motion of the leader, \( a_l \) is the semimajor axis of the leader orbit, and \( e_l \) is the orbit eccentricity. Differentiation of (22) results in the rate of change of the true anomaly,

\[
\ddot{\nu}(t) = -2n_l^2 e_l (1 + e_l \cos \nu(t))^3 \sin \nu(t) \tag{23}
\]

When the leader spacecraft is revolving the Earth in an elliptical orbit, the true anomaly rate \( \dot{\nu}(t) \) and true anomaly rate of change \( \ddot{\nu}(t) \) are bounded by constants, i.e. for all \( t \geq t_0 \geq 0 \)

\[
\alpha_\nu \leq \dot{\nu}(t) \leq \beta_\nu \quad \text{and} \quad |\ddot{\nu}(t)| \leq \beta_\dot{\nu} \tag{24}
\]

where \( \alpha_\nu, \beta_\nu, \beta_\dot{\nu} > 0 \). The latter is a reasonable standing assumption that we make.

IV. Controller Design

A. Preliminary results

We present first the non-adaptive case, i.e. when all parameters, bounds and disturbances are known. This result is briefly recalled here only for further development; see [7] for details. In that paper, under the assumptions that the values
of $\beta_\nu$ and $\beta_\nu$ on the leader true anomaly, as given in (24),
were known, the sum of forces working on the leader due
to control thrust and external perturbations was bounded, and
similar that the perturbation term was bounded, it was shown
that the controller structure
\[ u_f = -k_p e - k_d \dot{e} + n + mf \left( \frac{\mu}{r_f^2} I + \beta_\nu^2 D + \beta_\nu C \right) p \]
\[ + 2mf \beta_\nu \dot{C} \dot{p} + m_f \dot{p} \]
\[ \dot{p}_e = -a \theta \]
\[ \dot{\theta} = p_e + b e \]
where $e = p - p_\nu$, and $k_p$, $k_d$, $a$ and $b$ are sufficiently large
constants, results in a uniformly globally practically
asymptotically stable (UGPAS) closed loop system. The closed-loop
system may be written as
\[ m_f \ddot{e} = A(t, x) + G(t, e) \]
where
\[ A(t, x) := -C(\nu) \dot{e} - k_p e - k_d \dot{e} \]
and
\[ G(t, e) := G_1(t, e) + G_2(p_\nu(t), \dot{p}_\nu(t)), \]
consisting of the vanishing perturbation $G_1(t, e)$ and the
non-vanishing perturbation $G_2(p_\nu(t), \dot{p}_\nu(t))$, given by
\[ G_1 = mf(\delta_p D + \delta_\nu C)e \]
\[ G_2 = mf(\delta_p D + \delta_\nu C)p + 2mf \delta_\nu \dot{C} \dot{p} + F_d - \frac{m_f}{m_i} u \]
where $\delta_p = \beta_p - \nu$, $\delta_\nu = \beta_\nu - \nu$ and $\delta_\nu^2 = \beta_\nu^2 - \nu^2$. Using the
Lyapunov function
\[ V_1(x) = \frac{1}{2} x^T P_1 x \]
where $x = \left[ e^T, \dot{e}^T, \theta^T \right]^T$ is the state vector and
\[ P_1 := \begin{bmatrix} k_p & \varepsilon_1 m_f & 0 \\ \varepsilon_1 m_f & m_f & -\varepsilon_1 m_f \\ 0 & -\varepsilon_1 m_f & \frac{k_d}{b} \end{bmatrix} \]
with $\varepsilon_1 \geq 0$ as a design variable, it was shown that the
derivative along the trajectories of the closed-loop system is
\[ \dot{V}_1(x) \leq -\frac{1}{2} x^T Q_2(\nu) x + Q_0(\nu, p_\nu, \dot{p}_\nu) \]
where
\[ Q_2(\nu) = [q_{ij}], \quad i, j = 1, 2, 3 \]
with submatrices given by
\[ q_{11} = 2\varepsilon_1 (k_p I - m_f \delta_\nu^2 D) \]
\[ q_{12} = q_{21}^T = \varepsilon_1 C(\nu) - m_f (\delta_p D + \delta_\nu C) \]
\[ q_{22} = 2\varepsilon_1 m_f (b - 1) I \]
\[ q_{13} = q_{31}^T = \varepsilon_1 \left[ (k_d - k_p) I + m_f (\delta_p D - \delta_\nu C) \right] \]
\[ q_{33} = 2 \left( \frac{a}{b} - \varepsilon_1 \right) k_d I \]
\[ q_{23} = q_{32}^T = -\varepsilon_1 C(\nu) - \varepsilon_1 m_f a I \]
and
\[ Q_0 = \begin{bmatrix} \varepsilon_1 m_f [(\delta_p^2 + \delta_\nu) p + 2\delta_\nu \dot{p}] \\ -\varepsilon m_f [(\delta_p^2 + \delta_\nu) p + 2\delta_\nu \dot{p}] \end{bmatrix} \]
Furthermore, $\dot{V}_1(x)$ is negative definite if
\[ |x| \geq 2 \frac{q_0}{q_{2,m}} \]
where we have $q_0 \geq |Q_0(\nu(t), p_\nu(t), \dot{p}_\nu(t))|$ and $q_{2,m} \leq |Q_2(\nu(t))|$ for all $t \geq 0$, and that $q_{2,m}$ can be arbitrarily
enlarged by an appropriate choice of $k_p, k_d$ and $b$.

**B. Main result**

We are now ready to present our main result. We assume
that the sum of forces working on the leader due to control
thrust and external perturbations is bounded, such that $m_f \|f_d + u\| \leq \beta_1$, with $\beta_1 = [\beta_{1,e}, \beta_{1,d}, \beta_{1,h}]$, and
that the orbital perturbations are bounded as $\|f_d\| \leq \beta_2$, with $\beta_2 = [\beta_{2,e}, \beta_{2,d}, \beta_{2,h}]$ (cf. Sect. III-A). We assume that $\beta_1, \beta_1, \beta_2$ and $\beta_\nu$ are unknown and that the follower has available
measurements of relative position $p$ we have the following proposition:

**Proposition 6** Assuming that the desired relative position $p_\nu$,
desired relative velocity $\dot{p}_\nu$ and desired relative acceleration
$\ddot{p}_\nu$ are all bounded functions. Consider the control law given by (26), (27),
\[ u_f = -k_p e - k_d \dot{e} + n + m_f \left( \frac{\mu}{r_f^2} I + \beta_\nu^2 D + \beta_\nu C \right) p \]
\[ + 2m_f \beta_\nu \dot{C} \dot{p} + \beta_1 + m_f \dot{p} \]
where $e = p - p_\nu, k_p, k_d, a$ and $b$ are sufficiently large.
Consider the update law
\[ \dot{\theta} = H(t, x) = -\varepsilon_2 \Gamma B^T (e + p_\nu, \dot{p}_\nu) (e - \dot{\theta}) \]
where $\Gamma > 0$ satisfies $\gamma_m I \leq \Gamma \leq \gamma_M I, \varepsilon_2 \geq 0$, and
\[ B = [b_i], \quad i = 1, ..., 6 \]
where
\[ b_1 = D(e + p_\nu) \quad b_2 = 2Cp_\nu \quad b_3 = C(e + p_\nu) \]
\[ b_4 = n + \left( \frac{\mu}{r_f^2} I + \beta_\nu^2 D + \beta_\nu C \right)(e + p_\nu) \]
\[ b_5 = I \quad b_6 = -I \]
Define $\ddot{B}_0(t) := B(0, p_\nu(t), \dot{p}_\nu(t))$ and let $\gamma_M$ be sufficiently
small. Under these conditions and if moreover, $\ddot{B}_0$ is
persistently exciting, i.e. there exists $\mu, T > 0$ such that for all
$t \geq 0$,
\[ \int_t^{t+T} \ddot{B}_0(\tau) \ddot{B}_0(\tau) d\tau \geq \mu I, \]
the closed-loop system is weakly uniformly semiglobally prac-
tically asymptotically stable (USPAS).

**Remark 7** Notice that, in particular, Proposition 6 establishes
the convergence of all estimation errors to a ball that may be
arbitrarily diminished.
1) Proof of Proposition 6: The closed-loop dynamics of the system (14) and the controller structure (26)-(27), (45) are
\[ m_y \ddot{e} = A(t, x) + G(t, e) + B(e, p_*, p_*) \dot{\phi} \] (49)
with (29), (30), (47) and adaptation error $\dot{\phi} = \ddot{\phi} - \dot{\phi}$, with the parameter vector $\phi = \left[ \beta_0^p, \beta_0^p, \beta_0^p, \beta_1^p, \beta_1^p, \beta_2^p, \beta_2^p \right]^T$. Note that the matrices $A(t, x)$, $G(t, e)$ and $B(e, p_*, p_*)$ have the following properties: there exists positive constants $b_M, \rho_1, \rho_2$ and $\rho_3$ such that for all $t \geq 0$ and all $x \in \mathbb{R}^n$ we have that
\[ \max \left\{ \left| \dot{B}_0(t) \right|, \left| \frac{d}{dt} B_0(t) \right| \right\} \leq b_M \] (50)
and
\[ \left| B(e, p_*, p_*) - B_0(t) \right| \leq \rho_1 |e| \leq \rho_1 |x| \] (51)
\[ \|H(t, x)\| \leq \gamma_M \rho_2 |x|^2 + |x| \] (52)
\[ \|A(t, x) + G(t, e)\| \leq \rho_3 |x| \] (53)
where $\gamma_M$ is the largest eigenvalue of $\Gamma$ in (46). Further, for $Q_2(\dot{\nu}(t))$ defined in (36)-(42), we have that $\dot{q}_{2,m} \leq \dot{q}_{2,m}$ for all $t \geq 0$ and $q_{2,m}$ can be enlarged arbitrarily since all the controller gains are contained in $Q_2(\dot{\nu})$. However, the new controller structure results in
\[ Q_0 = \begin{bmatrix} \varepsilon_1 m_f & \delta_{p_2} + \delta_{p_3} + \delta_{p_4} + \delta_{p_5} & \delta_{p_6} + \delta_{p_7} + \delta_{p_8} \\ m_f & \delta_{p_2} + \delta_{p_3} + \delta_{p_4} + \delta_{p_5} & \delta_{p_6} + \delta_{p_7} + \delta_{p_8} \\ -\varepsilon_1 m_f & \delta_{p_2} + \delta_{p_3} + \delta_{p_4} + \delta_{p_5} & \delta_{p_6} + \delta_{p_7} + \delta_{p_8} \end{bmatrix}^T \] (54)
where $\delta_i = \beta_i - \frac{m}{m_f} (\Delta d_t + \Delta u_1)$ and $\delta_f = \beta_f - \Delta d_f$. Nevertheless, $Q_0(\dot{\nu}(t), \dot{p}_*(t), \dot{p}_*(t))$ is bounded, i.e. there exists a $q_0 \geq |Q_0(\dot{\nu}(t), \dot{p}_*(t), \dot{p}_*(t))|$. Hence, by defining $\chi = |x, \phi|^T$ and using $V_2(t, x) = V_1(t, x) + W(t, \chi)$ with
\[ W(t, \chi) = \frac{1}{2} \phi^T m_f \Gamma^{-1} \phi - \varepsilon_1 m_f e^T \dot{B}_0(t) \phi \]
\[ - \varepsilon_2 \int_{t}^{\infty} e^{(t-\tau)} \dot{B}_0(t) \phi^2 d\tau \] (55)
we find that
\[ \dot{V}_2 \leq - \frac{1}{4} q_{2, m} |x|^2 + q_0 |x| + \gamma_M \rho_1 |e| |\phi| + \frac{\gamma_M b_M}{2} |\phi|^2 \]
\[ + \frac{\gamma_M b_M}{2} |x|^2 + \varepsilon_2 b_M \rho_3 |x| |\phi| + \varepsilon_2 b_M \rho_1 |e| |\phi|^2 \]
\[ + 2 \varepsilon_2 b_M \rho_2 |\phi| |x|^2 |x^2 + |x|^2 - \varepsilon_2 e^{-T} \mu |\phi|^2 \]
\[ + \varepsilon_2 m_f b_M |e||\phi| + (\dot{\phi}^T + \gamma_M \rho_2 |x|^2 + \gamma_M \rho_2 |x|) \] (56)
Restrict the states to $|x| \leq \Delta$ and collect terms to obtain
\[ \dot{V}_2 \leq - \left[ \frac{1}{4} q_{2, m} - \frac{\gamma_M b_M}{2} - \frac{\gamma_M b_M}{2} \right] \Delta \]
\[ - \frac{3}{2} \Delta |x|^2 - \gamma_M \rho_1 |x|^2 + \frac{1}{2} \Delta \]
\[ + \frac{1}{2} \varepsilon_2 e^{-T} \mu |\phi|^2 \] (57)
where we have used $2|a| |b| \leq \lambda |a|^2 + \frac{1}{2} |b|^2$. Next, let $|\chi| \geq \sqrt{2} \delta$ then, $\Delta \geq \max \{ |\phi|, |x| \} \geq \delta$, i.e. either of the two hold: 1) $|x| \in [\delta, \Delta], |\phi| \in [0, \Delta]$, or 2) $|x| \in [0, \delta], |\phi| \in [\delta, \Delta]$. In the first case we have, defining $c = (40 + (2 + m_f) \varepsilon_2 b_M \gamma_M \rho_2 |\Delta|)$,
\[ [q_0 + (2 + m_f) \varepsilon_2 b_M \gamma_M \rho_2] \Delta \]
and in the second, $1 \leq |\phi|/\delta$ hence,
\[ [q_0 + (2 + m_f) \varepsilon_2 b_M \gamma_M \rho_2] \Delta \]
Setting $\varepsilon \leq 1/\Delta$ and collecting terms we have, for all $t \geq 0$ and all $\sqrt{2} \delta \leq |x| \leq \Delta$,
\[ \dot{V}_2(t, x) \leq - \frac{1}{4} q_{2, m} |x|^2 - \eta_1 (q_{2, m}, \gamma_M, \delta, \Delta) |x|^2 \]
\[ - \frac{1}{2} \varepsilon_2 e^{-T} \mu |\phi|^2 \] (58)
where
\[ \eta_1 := \frac{1}{4} q_{2, m} - \frac{\gamma_M b_M}{2} - \frac{\gamma_M b_M}{2} - \frac{1}{2} \Delta \]
\[ - \lambda_1 b_M \rho_3 - \lambda_2 b_M \rho_1 - \lambda_3 b_M \rho_2 \]
\[ - \frac{1}{2} \delta (\lambda_1 + 1) q_0 + (2 + m_f) b_M \gamma_M \rho_2 \] (59)
and
\[ \eta_2 := \frac{1}{4} q_{2, m} - \frac{\gamma_M b_M}{2} - \frac{1}{2} \Delta \]
\[ - \lambda_1 b_M \rho_3 - \lambda_2 b_M \rho_1 - \lambda_3 b_M \rho_2 \]
\[ - \frac{1}{2} \delta (\lambda_1 + 1) q_0 + (2 + m_f) b_M \gamma_M \rho_2 \] (60)
Making $\lambda_1$ with $i \in \{1, \ldots, 4\}$ and $q_{2,m}$ large and imposing that $\gamma_M$ be small enough to satisfy
\[ \gamma_M \leq \varepsilon_2 e^{-T} \frac{\mu}{b_M} \] (61)
we obtain that for all $t \geq 0$ and all $\sqrt{2} \delta \leq |x| \leq \Delta$.
\[ \dot{V}_2(t, x) \leq - \frac{1}{4} q_{2, m} |x|^2 - \frac{1}{2} \varepsilon_2 e^{-T} \mu |\phi|^2 \] (62)
Indeed, we can now choose
\[ q_{2, m} = 2 c_0 + 4 c_1 \Delta + 2 \frac{c_0}{\delta} \] (63)
where
\[ c_0 = \gamma_M b_M - 2 (2 + m_f) \gamma_M b_M \rho_2 + \gamma_M b_M \rho_1 + \lambda_3 b_M ^2 \] (64)
\[ c_1 = \gamma_M \rho_1 \] (65)
\[ c_2 = \lambda_1 b_M \rho_3 + 2 (\lambda_1 + 1) (q_0 + (2 + m_f) b_M \gamma_M \rho_2) \] (66)
with the use of $\rho_3 = \rho_3 \Delta/\delta$ to show the dependency of the terms in $\rho_3$ on $\delta$ and $\Delta$ through (29). Hence, from Corollary 5 we conclude that the system (49),(46), is weakly USPAS.

Remark 8 Note that the bounds $\beta_0^p, \beta_0^p, \beta_0^p, \beta_0^p, \beta_0^p, \beta_0^p, \beta_0^p$ are used as constant approximations of the corresponding time-varying functions in the controller structure. If such constant approximations were to be used directly in the controller without adaptation, it would be better to choose some more appropriate average values instead of bounds. However, when adaptation is used, the adaptation parameters will converge to
a ball around such average constant approximations.

**Remark 9** Technically, from the proof above one cannot conclude that the system is USPAS (i.e. that we can take \( \delta^* = 0 \)) unless a restriction is imposed on \( \mu \) and \( T \), equivalently, on the follower reference trajectories. In other words, it is possible to obtain an arbitrary parameter estimation error only for rich reference signals; which is quite reasonable.

V. SIMULATIONS

In this section, simulation results for a spacecraft formation are presented. The leader spacecraft is assumed to be following an elliptic orbit with eccentricity \( e = 0.6 \). Both spacecraft have mass \( m = 100 \) kg. The follower is assumed to have available continuous thrust in all directions, limited to \( 27 \) N. The follower has initial values \( p = [20, 10, -20]^T \), and is further commanded to track sinusoidal trajectories around the leader, given as

\[
p_f(t) = \begin{bmatrix}
-10 \cos\left(\frac{3\pi}{T_0} t\right), \\
10 \sin\left(\frac{4\pi}{T_0} t\right), \\
5 \cos\left(\frac{5\pi}{T_0} t\right)
\end{bmatrix}^T
\]

where \( T_0 \) is the orbital period. A possible scenario for this motion is in-orbit inspection, where the follower moves in orbit around the leader. All adaptation parameters have initial values 0, except for the follower spacecraft mass, which is assumed to be known with a 30\% accuracy. In all simulations performed, we used the controller gains \( k_p = 4 , k_d = 6 \), \( a = 1 \) and \( b = 5 \). In the adaptive controller we used \( \gamma_2 = 0.3 \) and \( \Gamma = \text{diag}([\gamma_i]) \), where \( \gamma_1 = 3.3 \cdot 10^{-6} \), \( i = 1, 3, 8, 9 \), \( \gamma_2 = 6.5 \cdot 10^{-4} \), \( \gamma_4 = 6.7 \cdot 10^2 \), \( \gamma_5 \) = \( \gamma_6 = 3.3 \cdot 10^{-7} \) and \( \gamma_7 \) = \( \gamma_10 = 3.3 \cdot 10^{-8} \). Orbital perturbations forces due to gravity and aerodynamic drag are included in the simulations.

A. Results

Fig. 1 shows the performance of the adaptive control law (26),(27) and (45) with the parameter update law (46). The follower settles at the desired trajectory and proceeds to track the trajectory. The USPAS property of the closed-loop system is seen in the figure as persistent oscillations around the origin. Note that the magnitude of the oscillations can be arbitrarily diminished by increasing the controller gains. The parameter adaptation can be seen in Fig. 2.

VI. CONCLUSION

We have presented a solution to the problem of tracking relative translation in a leader-follower spacecraft formation using feedback from relative position only, using an approximate differentiation filter to provide sufficient knowledge about the velocity error. The solution uses adaptation to find constant approximate values for orbital perturbations and leader control force, together with follower spacecraft mass, and leader true anomaly rate and rate of change. The resulting closed-loop system was proved to be weakly uniformly semiglobally practically asymptotically stable (USPAS).

REFERENCES